LAU-Mathematics Tournament 2017 **Spring 2017**

Topic: Analysis **Date: May 20, 2017**

Duration: 45 min

Grade	Duration

1. Suppose that $f : [0,1] \to \mathbb{R}$ is continuous. Compute $\lim_{x\to 0^+} \int_{t=0}^{1} \frac{xf(t)}{x^2+t^2}\, dt$.

2. Let $f : [0,1] \to \mathbb{R}$ a continuous function. Show that $\int_{t=0}^{1} f(t^n)\, dt \to f(0)$ when $n \to \infty$.

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3. Set $u_n(x) = (-1)^n \ln\left(1 + \frac{x}{n(1+x)}\right)$. Study the normal convergence of the series $\sum u_n$.

4. For $x > 0$ show that $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+x} = \int_{t=0}^{1} \frac{t^{x-1}}{t+1}\, dt$.

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5. Evaluate the limit $\lim\limits_{n\to\infty} \left(\dfrac{n^2+n+3}{n^2+3n+5}\right)^n$.

6. Give two equivalent real sequences u_n and v_n such that $\sum u_n$ is convergent and $\sum v_n$ is divergent.

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7. Show that the power series $\sum\limits_{n=0}^{+\infty} \left(1+\dfrac{1}{n}\right)^{n^2} x^n$ converges if $\dfrac{-1}{e} < x < \dfrac{1}{e}$.

8. Calculate the following limit:

$$\lim_{x\to 0}\left(\frac{e^x - e^{-x} - 2\sin x}{x(1-\cos x)}\right)$$

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9. Let w satisfy $\mathrm{Re}(w) < 0$ and let γ be the straight line segment joining 0 to w. Compute $\int_\gamma e^z dz$ and deduce that $|e^w - 1| \leq |w|$.

10. Let $f \in C^1([a,b],\mathbb{R})$ such that $f(a) = 0$. Show that

$$\int_a^b |f(t)|^2 dt \leq \frac{(b-a)^2}{2}\int_a^b |f'(t)|^2 dt.$$

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11. Consider two holomorphic functions $f_1(z)$ and $f_2(z)$ defined on an open connected $\mathcal{U}$ of $\mathbb{C}$. Suppose that for every $z \in \mathcal{U}$

$$f_1(z) + \overline{f_2(z)} = 0.$$

Show that f_1 and f_2 are constants on $\mathcal{U}$.

12. Let $r = r(\theta)$, $a \leq \theta \leq b$, describe a plane curve in polar coordinates. Show that the arc Length of this curve is given by

$$\int_a^b [r^2 + (r')^2]^{1/2} d\theta$$

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13. Let g be a differentiable function on (a, b) containing 0. Suppose that

    (a) $g(x+y) = \frac{g(x)+g(y)}{1-g(x)g(y)}$ for all x, y and $x + y$ in (a, b).

    (b) $\lim_{h\to 0} g(h) = 0$.

    (c) $\lim_{h\to 0} \frac{g(h)}{h} = 1$.

    Show that $g(0) = 0$ and $g'(x) = 1 + g^2(x)$. Find then $g(x)$

14. The function $f(x)$ has the property that, for some real positive constant c, the expression

$$\frac{f^{(n)}(x)}{n + x + c}$$

is independent of n for all nonnegative integers n, provided that $n+x+c \neq 0$. Given that $f'(0) = 1$ and $\int_0^1 f(x)dx = c + e - 2$, determine the value of c.

15. The real-valued infinitely differentiable function $f(x)$ is such that $f(0) = 1$, $f'(0) = 2$, and $f''(0) = 3$. Furthermore, f has the property that

$$f^{(n)}(x) + f^{(n+1)}(x) + f^{(n+2)}(x) + f^{(n+3)}(x) = 0,$$

for all $n \geq 0$, where $f^{(n)}(x)$ denotes the n^{th} derivative of f. Find $f(x)$.

MARKS : 1. 3 pts for each exercise.

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1. Soit $f : [0, 1] \to \mathbb{R}$ continue. Chercher $\lim_{x\to 0^+} \int_{t=0}^{1} \frac{xf(t)}{x^2+t^2}\, dt$.
2. Soit $f : [0, 1] \to \mathbb{R}$ continue. Montrer que $\int_{t=0}^{1} f(t^n)\, dt \to f(0)$ lorsque $n \to \infty$.

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3. Soit $u_n(x) = (-1)^n \ln\left(1 + \frac{x}{n(1+x)}\right)$. Etudier la convergence normale de la série $f(x)$ converge $\sum u_n$.
4. Montrer, pour $x > 0$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+x} = \int_{t=0}^{1} \frac{t^{x-1}}{t+1}\, dt$.

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5. Trouver la limite $\lim\limits_{n\to\infty} \left(\frac{n^2+n+3}{n^2+3n+5}\right)^n$.
6. Donner deux suites réelles équivalentes u_n et v_n, tel que $\sum u_n$ est convergente et $\sum v_n$ est divergente.

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7. show that the power series $\sum\limits_{n=0}^{+\infty} \frac{(2n)!}{(n!)^2} x^n$ converge if $\frac{-1}{4} < x < \frac{1}{4}$.
8. Calculer la limite suivante :

$$\lim_{x\to 0} \left(\frac{e^x - e^{-x} - 2\sin x}{x(1-\cos x)}\right)$$

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9. Soit w tel que $\mathrm{Re}(w) < 0$ et soit γ le segment de droite d'extrémités 0 et w. Calculer $\int_\gamma e^z dz$. En déduire que $|e^w - 1| \leq |w|$.

10. Soit $f \in C^1([a,b],\mathbb{R})$ tel que $f(a) = 0$. Montrer que

$$\int_a^b |f(t)|^2 dt \leq \frac{(b-a)^2}{2} \int_a^b |f'(t)|^2 dt$$

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11. Soit $f_1(z)$ et $f_2(z)$ deux fonctions holomorphes définies sur un ouvert connexe $\mathcal{U}$ de $\mathbb{C}$. On suppose que pour tout $z \in \mathcal{U}$

$$f_1(z) + \overline{f_2(z)} = 0.$$

Montrer que f_1 et f_2 sont constantes sur $\mathcal{U}$.

12. Soit $r = r(\theta)$, $a \leq \theta \leq b$, l'équation d'une courbe en coordonnées polaires. Montrer que la longueure de cette courbe est donnée par

$$\int_a^b [r^2 + (r')^2]^{1/2} d\theta$$

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13. On considre une fonction différentiable sur un interval $]a,b[$ contenant 0. On suppose que

   (a) $g(x+y) = \frac{g(x)+g(y)}{1-g(x)g(y)}$ pour tout x, y et $x+y$ dans $]a,b[$.

   (b) $\lim_{h\to 0} g(h) = 0$.

   (c) $\lim_{h\to 0} \frac{g(h)}{h} = 1$.

   Montrez que $g(0) = 0$ et $g'(x) = 1 + g^2(x)$. Trouvez $g(x)$

14. Une fonction f a la propriété que, pour une constante reélle $c > 0$, l'expression

$$\frac{f^{(n)}(x)}{n+x+c}$$

   est indépendante de n pour tout $n \in \mathbb{N}$ sachant que $n + x + c \neq 0$. On donne $f'(0) = 1$ et $\int_0^1 f(x)dx = c + e - 2$, déterminer la valeur de c.

15. Une fonction f infiniment différentiable satisfait $f(0) = 1$, $f'(0) = 2$ et $f''(0) = 3$. En plus, f satisfait

$$f^{(n)}(x) + f^{(n+1)}(x) + f^{(n+2)}(x) + f^{(n+3)}(x) = 0,$$

   pour tout $n \geq 0$. Ici $f^{(n)}(x)$ est la derivée d'ordre n de f. Trouvez f.

BARÈME : 1. 3 pts pour chaque exercice.

LAU-Mathematics Tournament 2017 **Spring 2017**

Topic: Algebra **Date: May 20, 2017**

Duration: 45 min

Grade	Duration

1. Solve in $\mathbb{Z}$, the following system:

$$(S)\begin{cases} x \equiv 6 \mod(17) \\ x \equiv 4 \mod(15) \end{cases}$$

2. Let $x \in \mathbb{R}^*$ such that $x + \dfrac{1}{x} \in \mathbb{Z}$. Prove that $x^n + \dfrac{1}{x^n} \in \mathbb{Z}$ for all $n \in \mathbb{N}$.

3. Find A^n, where $A = \begin{pmatrix} 2 & 0 & 1 & 3 \\ -4 & 0 & -2 & -6 \\ 6 & 0 & 3 & 9 \\ -8 & 0 & -4 & -12 \end{pmatrix}$

4. For $n \geq 2$ a fixed number, we denote by E the vector space of all the polynomials with coefficients in $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and with degree less or equals to n. We define the function f from E to E by $f(P) = P - P'$, for all P in E.

   (a) Prove that f is an non-diagonalizable isomorphism.

   (b) Let $Q \in E$. Find $P \in E$ such that $f(P) = Q$.

MARKS : 1. [8] **2.** [11] **3.** [11] **4.** [15]

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1. Résoudre dans $\mathbb{Z}$ le système suivant:

$$(S)\begin{cases} x \equiv 6 \mod(17) \\ x \equiv 4 \mod(15) \end{cases}$$

2. Soit $x \in \mathbb{R}^*$ tel que $x + \dfrac{1}{x} \in \mathbb{Z}$. Montrer que $x^n + \dfrac{1}{x^n} \in \mathbb{Z}$ pour tout $n \in \mathbb{N}$.

3. Trouver A^n, si $A = \begin{pmatrix} 2 & 0 & 1 & 3 \\ -4 & 0 & -2 & -6 \\ 6 & 0 & 3 & 9 \\ -8 & 0 & -4 & -12 \end{pmatrix}$.

4. Pour $n \geq 2$ un nombre fixé, on note par E l'espace vectoriel des polynômes à coefficients dans $\mathbb{R}$ de degré inférieur ou égal à n. On définit l'application f de E dans E par $f(P) = P - P'$ pour tout $P \in E$.

   (a) Montrer que l'application f est un isomorphisme non-diagonalisable.

   (b) Soit $Q \in E$. Trouver $P \in E$ tel que $f(P) = Q$.

BARÈME : 1. [8] **2.** [11] **3.** [11] **4.** [15]

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LAU-Mathematics Tournament 2017

Spring 2017

Topic: Combinatorics

Date: May 20, 2017

Duration: 30 min

Grade	Duration

You may use mathematical objects to provide your answers. Final numerical answers are not required.

1. Suppose a computer password consists of eight to ten letters and/or digits. How many different possible passwords are there? Remember that an upper-case letter is different from a lower-case one.

2. In the game of "Hollywood squares," X's and O's may be placed in any of the nine squares of a tic-tac-toe board (a 3×3 matrix) in any order. Squares may also be blank, i.e., not containing either an X or and O. How many different boards are there?

3. Three dice of different colors are loaded in a way that an even number occurs three times more often than an odd one. Find the probability that the sum of the numbers that appear on the top faces of the dice when tossed is at least 7, given that one of them shows a 4.

4. In how many ways can 9 persons stand in a row if 3 out of them insist on following each other?

5. In a game of 52 cards, a hand consists of 5 cards. Find the probability that a person holds an Ace and at least 1 Queen.

6. A palindromic number is a number that remains the same when its digits are reversed (for example 2442). How many palindromic numbers are there between 1000 and 1000?

7. There are 97 men and 3 women in an organization. A committee of 5 people is chosen at random, and one is randomly designated as chairperson. What is the probability that the committee includes all 3 women and has one of the women as chairperson?

8. Let n be an integer. What is the probability that an application chosen at random from the set $\{1, \ldots, n\}$ into its self is a bijective?

9. Consider 3 integers m, n et k with $k \leq m$ et $k \leq n$. Give the coefficient of x^k in the expression $(1+x)^m(1+x)^n$ and Justify that : $\sum_{i=0}^{k} C_m^i C_n^{k-i} = C_{m+n}^k$

10. We have n cages, distributed circularly, and k lions, so ferocious that two of these beasts must be separated by an empty cage at least. Let L_n^k be the number of ways to distribute the large cats in the cages. Find L_0^k for $k \geq 1$, L_n^1 and L_n^2 for $n \geq 1$

Marks: 3 points for each question

1. Suppose a computer password consists of eight to ten letters and/or digits. How many different possible passwords are there? Remember that an upper-case letter is different from a lower-case one.

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